Buckingham pi Theorem for Turbomachinery Aerodynamic Performance

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In general, turbomachinery aero(or hydro)dynamic performance parameters of,

- Total enthalpy change — Δh_o $[L^2T^{-2}]$
- Power — P_w $[ML^2T^{-3}]$

are dependent of,

- Inlet total pressure ┐ → Reduces to
- Inlet total temperature ┘ → Inlet total density — ρ_{o1} $[ML^{-3}]$
- Rotational speed — N $[T^{-1}]$
- Mass flow rate — $\dot{m}$ $[MT^{-1}]$
- Machine diameter — D $[L]$
- Fluid dynamic viscosity — μ $[ML^{-1}T^{-1}]$
- Fluid compressibility → Inlet sonic velocity — a_{o1} $[LT^{-1}]$

It means that total 8 dimensional variables govern performance phenomena in physics. When 3 variables of {D, N, ρ_{o1}} are chosen as primary, the problem reduces to only 5 dimensionless (pi) variables.

Subscript
o : Total state
1 : Inlet, 2 : Outlet

$$\Pi_1 = D^a N^b \rho_{o1}^c \cdot \Delta h_o$$

Unit arrangement gives,

$$L^a T^{-b} M^c L^{-3c} \cdot L^2 T^{-2} = M^0 L^0 T^0$$

$$\therefore a = b = -2, \qquad c = 0$$

$$\Pi_1 = \frac{\Delta h_o}{(ND)^2}$$ Work (or Head) Coefficient

$$\Pi_2 = D^a N^b \rho_{o1}^c \cdot P_w$$

Unit arrangement gives,

$$L^a T^{-b} M^c L^{-3c} \cdot M L^2 T^{-3} = M^0 L^0 T^0$$

$$\therefore a = -5, \qquad b = -3, \qquad c = -1$$

$$\Pi_2 = \frac{P_w}{\rho_{o1} N^3 D^5}$$ Power Coefficient

$$\Pi_3 = D^a N^b \rho_{o1}^c \cdot \dot{m}$$

Unit arrangement gives,

$$L^a T^{-b} M^c L^{-3c} \cdot M T^{-1} = M^0 L^0 T^0$$

$$\therefore a = -3, \qquad b = c = -1$$

$$\Pi_3 = \frac{\dot{m}}{\rho_{o1} N D^3}$$ Flow Coefficient

$$\Pi_4 = D^a N^b \rho_{o1}^c \cdot a_{o1}$$

Unit arrangement gives,

$$L^a T^{-b} M^c L^{-3c} \cdot L T^{-1} = M^0 L^0 T^0$$

$$\therefore a = b = -1, \qquad c = 0$$

$$\Pi_4 = \frac{a_{o1}}{ND}$$ Speed Coefficient $^{-1}$

$\Pi_5 = D^a N^b \rho_{o1}^c \cdot \mu$

Unit arrangement gives,

$L^a T^{-b} M^c L^{-3c} \cdot M L^{-1} T^{-1} = M^0 L^0 T^0$

$\therefore a = -2, \quad b = c = -1$

$\Pi_5 = \dfrac{\mu}{\rho_{o1} N D^2}$ Reynolds Number $^{-1}$

Therefore, a general relation of dimensionless variables describing turbomachinery aero (hydro) dynamic performance is,

$$\frac{\Delta h_o}{(ND)^2}, \eta, \frac{P_w}{\rho_{o1} N^3 D^5} = f\left\{\frac{\dot{m}}{\rho_{o1} N D^3}, \frac{\rho_{o1} N D^2}{\mu}, \frac{ND}{a_{o1}}\right\} \tag{1}$$

Now it is clear how the equation (2.6a) of Dixon's book was derived, noting that the isentropic enthalpy change was selected rather than the current actual one. The specific heat ratio (γ) needs to drop out, however, because an ideal gas was not assumed yet.

$$\frac{\Delta h_{0s}}{N^2 D^2}, \eta, \frac{P}{\rho_{01} N^3 D^5} = f\left\{\frac{\dot{m}}{\rho_{01} N D^3}, \frac{\rho_{01} N D^2}{\mu}, \frac{ND}{a_{01}}, \gamma\right\}. \tag{2.6a}$$

S.L. Dixon and C.A. Hall, Fluid Mechanics and Thermodynamics of Turbomachinery, 6th Edition, 2010

Because the equation (1) serves a general case, there is no need for separately deriving dimensionless variables for incompressible fluids where the speed of sound goes infinite at a constant density. Equation (1) thus reduces to, without the speed coefficient for an incompressible fluid,

$$\frac{gH}{(ND)^2}, \eta, \frac{P_w}{\rho_{o1} N^3 D^5} = f\left\{\frac{Q}{ND^3}, \frac{\rho_{o1} N D^2}{\mu}\right\} \tag{2}$$

where *H, Q* and *g* denote total head, inlet volumetric flow and gravity, respectively.

In other words, ***when fluid compressibility cannot be neglected, the speed coefficient is required for a correct normalization of performance.***

Because turbomachinery work is directly related with the change of angular momentum, the work (or head) coefficient, expressed by ψ, is normalized using the blade speed ($U = r\omega$) which characterizes the machine of interest. For compressors or pumps the work (or head) coefficient becomes,

$$\frac{\Delta h_o}{(ND)^2} = \frac{\Delta h_o}{U_2^2} \qquad \textit{for compressible fluids} \qquad (3)$$

$$= \frac{gH}{U_2^2} \qquad \textit{for incompressible fluids} \qquad (4)$$

However, in the pump inducer where cavitation onset is considered critical, inlet tip speed has been used instead.

$$\frac{\Delta h_o}{(ND)^2} = \frac{gH}{U_{1,tip}^2} \qquad (5)$$

Power coefficient is likewise expressed by, using blade speed (*U*) and radius (*r*),

$$\frac{P_w}{\rho_{o1} N^3 D^5} = \frac{P_w}{\rho_{o1} U_2^3 r_2^2} \qquad (6)$$

Flow coefficient, expressed by ϕ, is usually considered at the inlet,

$$\frac{\dot{m}}{\rho_{o1} N D^3} = \frac{\dot{m}}{\rho_{o1}(ND)D^2} = \frac{\dot{m}}{\rho_{o1} a_{o1} d_{1,tip}^2} \qquad \textit{for compressible fluids} \qquad (7)$$

$$\frac{Q}{ND^3} = \frac{C_{m1}}{U_{1,tip}} \qquad \textit{for incompressible fluids} \qquad (8)$$

where d and C_m denote diameter and meridional velocity, respectively.

Speed coefficient in compressible fluids is also called machine Mach number, expressed by M_U, usually characterized by the blade speed which will best represent the machine.

$$\frac{ND}{a_{o1}} = \frac{U_2}{a_{o1}} \qquad (9)$$

Reynolds number of turbomachinery has been with different variables depending on the problem of concern, but it would be suggested as, considering the selection of inlet total density in the current analogy,

$$\frac{\rho_{o1} N D^2}{\mu} = \frac{\rho_{o1} C_{m1} l_c}{\mu_1} \qquad (10)$$

where l_c is blade chord length at midspan.

Dimensionless performance maps of turbomachinery are plotted using variables of eq. (1) for compressible fluids, or those of eq. (2) for incompressible fluids. One example of the head (or work) coefficient would be as illustrated in Fig.1, for compressors and pumps.

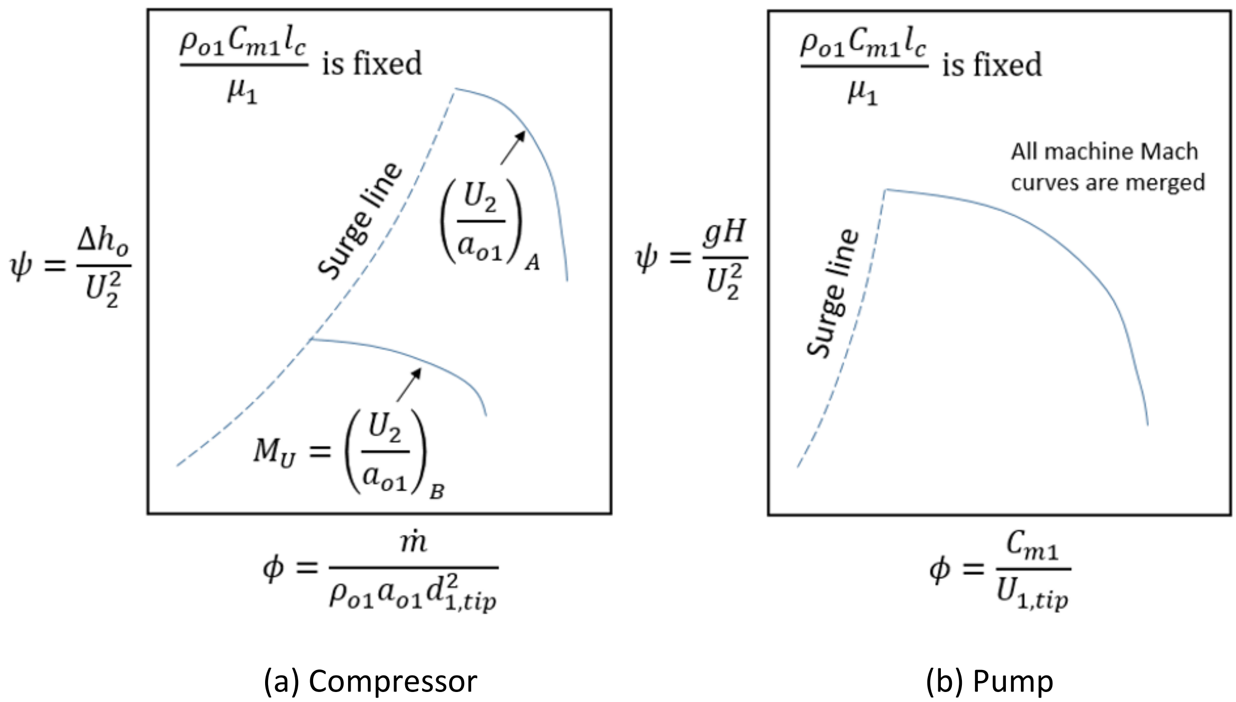

Fig.1 Concepts of dimensionless performance maps

In general, there are two different groups, in terms of the way of presenting normalized performance maps, depending on historical backgrounds of industry, even though they originate from the Buckingham pi theorem.

(a) Flow Coefficient / Work(or Head) Coefficient / Efficiency, per **Machine Mach**
(b) Corrected Flow / Corrected Pressure Ratio / Efficiency, per **Corrected Speed**

Group (a) is what we have talked about so far.

Group (b) prefers pressure ratio and speed to work (or head) coefficient and machine Mach number in compressible map readings, allowing some tweaks of creating a so-called "corrected" variable from non-dimensional parameters. Historically Group (a) has been actively used in some of gas industry including chillers, while ***Group (b) has been much more popular in general turbomachinery industry***, thanks to its convenient definitions. NASA used to call the corrected variable "equivalent" one instead.

To simplify the case, before moving on to the details of Group (b), the ideal gas is assumed. *Please note there is no limitation in Group (b) either when it comes to a real gas.*

Machine Mach number can be reduced to the corrected speed as, *when an identical machine is considered with an identical gas but working at different operating conditions*.

$$\frac{ND}{a_{o1}} = \frac{U_2}{a_{o1}} = \frac{r_2\frac{2\pi n}{60}}{\sqrt{\gamma R T_{o1}}} \quad \rightarrow \quad \frac{n}{\sqrt{T_{o1}}} \tag{11}$$

where *R* and *n* are gas constant and rotational speed in rpm, respectively.

In the similar manner, the flow coefficient is reduced to the corrected mass flow as,

$$\frac{\dot{m}}{\rho_{o1}ND^3} = \frac{\dot{m}}{\rho_{o1}a_{o1}d_{1,tip}^2} = \frac{\dot{m}}{\frac{p_{o1}}{RT_{o1}}\sqrt{\gamma RT_{o1}}d_{1,tip}^2} = \frac{\dot{m}\sqrt{RT_{o1}}}{p_{o1}\sqrt{\gamma}d_{1,tip}^2} \quad \rightarrow \quad \frac{\dot{m}\sqrt{T_{o1}}}{p_{o1}} \tag{12}$$

Accordingly the power coefficient will be reduced to the corrected power as,

$$\frac{P_W}{\rho_{o1}N^3D^5} = \frac{P_W}{\rho_{o1}U_2^3r_2^2} = \frac{P_W}{\rho_{o1}r_2^5\left(\frac{2\pi n}{60}\right)^3} \quad \rightarrow \quad \frac{P_W}{\rho_{o1}\left(n_c\sqrt{T_{o1}}\right)^3} \tag{13}$$

where n_c is the corrected speed (rpm) of eq. (11).

Physical pressure ratio, p_{o2}/p_{o1}, which might be inattentively considered good enough to be used with eq. (11) to eq. (13) because it is also dimensionless, has been loosely accepted as it is, for the purpose of easy engineering. However, ***to be perfectly valid in physics, it should be converted to the corrected pressure ratio*** through an actual change of total enthalpy and isentropic efficiency. That is because the change of enthalpy in turbomachinery is governed by the change of temperature and pressure through efficiency.

Eq.(3) can be reduced to, for an ideal gas,

$$\frac{\Delta h_o}{U_2^2} = \frac{c_pT_{o1}\left(\frac{T_{o2}}{T_{o1}}-1\right)}{U_2^2} = \frac{c_pT_{o1}\left\{\left(\frac{p_{o2}}{p_{o1}}\right)^{\frac{\gamma-1}{\gamma}}-1\right\}}{\eta U_2^2}$$

Assuming a constant isentropic efficiency (as for now), the relation between the physical and the corrected pressure ratio is as follows.

$$\left[\frac{c_pT_{o1}\left\{\left(\frac{p_{o2}}{p_{o1}}\right)^{\frac{\gamma-1}{\gamma}}-1\right\}}{U_2^2}\right]_{physical} = \left[\frac{c_pT_{o1}\left\{\left(\frac{p_{o2}}{p_{o1}}\right)^{\frac{\gamma-1}{\gamma}}-1\right\}}{U_2^2}\right]_{corrected} \tag{14}$$

Eq. (14) implies that unless the change of inlet total temperature, gas properties and blade speed remains negligible between physical and corrected conditions, the corrected pressure ratio should be used in non-dimensional maps. By the same analogy, it is to be applied for turbines as well.

Despite being dimensional as a result (except for eq. (14)), eq. (11) to eq. (14) serve as corrected performance parameters, considering the effects of different operating conditions with an ideal gas. One typical example of the compressor corrected map can be shown in Fig.2, when an identical machine is considered with an identical gas but working at different operating conditions. It can be frequently found elsewhere, but again, to be exactly valid in physics, the corrected pressure ratio, from eq. (14), should be used there.

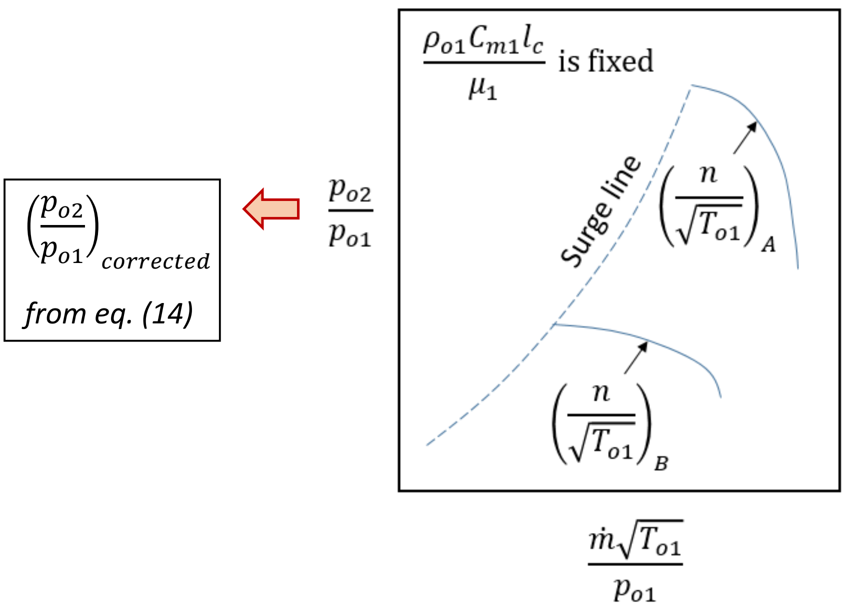

Fig.2 Ideal-gas compressor corrected map in an identical machine and working gas

Sometimes there is a case in industrial applications where the ideal gas assumption is still valid, but gas constants and specific heat ratios vary at different operating points in a gas mixture. It would be more convenient to consider performance parameters corrected to the standard inlet of air, for example, once each value of representative gas constant and specific heat ratio of the mixture at all of operating points is found. In that case, for an identical machine, eq.(11) to (14) should be rewritten as, Corrected speed, n_c (rpm), can be found from, $\frac{ND}{a_{o1}} = \frac{U_2}{a_{o1}} = \frac{r_2 \frac{2\pi n}{60}}{\sqrt{\gamma R T_{o1}}}$,

$$\left(\frac{n}{\sqrt{\gamma R T_{o1}}}\right)_{physical} = \left(\frac{n}{\sqrt{\gamma R T_{o1}}}\right)_{corrected} \tag{11b}$$

Corrected mass flow can be found from, $\frac{\dot{m}}{\rho_{o1} N D^3} = \frac{\dot{m}}{\rho_{o1} a_{o1} d_{1,tip}^2} = \frac{\dot{m}\sqrt{R T_{o1}}}{p_{o1}\sqrt{\gamma}\, d_{1,tip}^2}$,

$$\left(\frac{\dot{m}\sqrt{R T_{o1}}}{p_{o1}\sqrt{\gamma}}\right)_{physical} = \left(\frac{\dot{m}\sqrt{R T_{o1}}}{p_{o1}\sqrt{\gamma}}\right)_{corrected} \tag{12b}$$

Corrected power can be found from, $\frac{P_W}{\rho_{o1} N^3 D^5} = \frac{P_W}{\rho_{o1} U_2^3 r_2^2} = \frac{P_W}{\rho_{o1} r_2^5 \left(\frac{2\pi n}{60}\right)^3}$,

$$\left(\frac{P_W}{\rho_{o1} n^3}\right)_{physical} = \left(\frac{P_W}{\rho_{o1} n^3}\right)_{corrected} \tag{13b}$$

Corrected pressure ratio can be found from,

$$\left[\frac{c_p T_{o1}\left\{\left(\frac{p_{o2}}{p_{o1}}\right)^{\frac{\gamma-1}{\gamma}} - 1\right\}}{n_2^2}\right]_{physical} = \left[\frac{c_p T_{o1}\left\{\left(\frac{p_{o2}}{p_{o1}}\right)^{\frac{\gamma-1}{\gamma}} - 1\right\}}{n_2^2}\right]_{corrected} \tag{14b}$$

Corrected parameter definitions, eq. (11) to (14), are assuming the isentropic efficiency is unchanged, valid only when the other non-dimensional variable of the Reynolds number, eq. (10), is conserved between physical and corrected conditions. If there is a significant gap in the two Reynolds numbers, an additional correction for the efficiency is required as called Reynolds number correction. It has been generally built from empirical correlations per machine type. More corrections are necessary if any factor, not considered in the map normalization, comes into play, such as rotor tip clearance effects.

For a real gas, Group (b) expressions need to be generalized, without using gas constant or gas specific heat ratio. It means we stick to eq. (6), eq. (7) and eq. (9) as, for an identical machine,

$$\frac{\dot{m}}{\rho_{o1} a_{o1} d_{1,tip}^2} \quad \rightarrow \quad \frac{\dot{m}}{\rho_{o1} a_{o1}} \tag{15}$$

$$\frac{ND}{a_{o1}} \quad \rightarrow \quad \frac{n}{a_{o1}} \tag{16}$$

$$\frac{P_w}{\rho_{o1} N^3 D^5} \quad \rightarrow \quad \frac{P_w}{\rho_{o1} n^3} \tag{17}$$

For the corrected pressure ratio, eq. (3) reduces to,

$$\frac{\Delta h_o}{(ND)^2} \quad \rightarrow \quad \frac{h_{o2}(p_{o2}, T_{o2}) - h_{o1}(p_{o1}, T_{o1})}{n^2} \tag{18}$$

Introducing a reference state for the correction, atmospheric pressure of 1 atm and ambient temperature of 293.15 K are generally considered, but they can be changed as long as they are specified in the plot. For example, the aerospace standard temperature of 288.15 K might not be used for some gases, if they become liquid at the temperature and 1 atm.

Similarly, *for a real gas in an identical machine*, corrected mass flow, corrected speed, corrected power and corrected pressure ratio are defined through eq. (15) to eq. (18),

$$\begin{aligned}
\left(\frac{\dot{m}}{\rho_{o1} a_{o1}}\right)_{physical} &= \left(\frac{\dot{m}}{\rho_{o1} a_{o1}}\right)_{corrected} \\
\left(\frac{n}{a_{o1}}\right)_{physical} &= \left(\frac{n}{a_{o1}}\right)_{corrected} \\
\left(\frac{P_w}{\rho_{o1} n^3}\right)_{physical} &= \left(\frac{P_w}{\rho_{o1} n^3}\right)_{corrected} \\
\left(\frac{h_{o2}(p_{o2}, T_{o2}) - h_{o1}(p_{o1}, T_{o1})}{n^2}\right)_{at\ physical\ \frac{p_{o2}}{p_{o1}}} &= \left(\frac{h_{o2}(p_{o2}, T_{o2}) - h_{o1}(p_{o1}, T_{o1})}{n^2}\right)_{at\ corrected\ \frac{p_{o2}}{p_{o1}}}
\end{aligned} \tag{19}$$

Therefore, *for a real gas in an identical machine*,

$$\dot{m}_{corrected} = \dot{m}_{physical} \times \frac{(\rho_{o1} a_{o1})_{corrected}}{(\rho_{o1} a_{o1})_{physical}} \tag{20}$$

$$n_{corrected} = n_{physical} \times \frac{(a_{o1})_{corrected}}{(a_{o1})_{physical}} \tag{21}$$

$$P_{w_{corrected}} = P_{w_{physical}} \times \frac{(\rho_{o1} n^3)_{corrected}}{(\rho_{o1} n^3)_{physical}} \tag{22}$$

$$\left(\frac{p_{o2}}{p_{o1}}\right)_{corrected} \quad \textit{is found from eq. (19) through trial and error.} \tag{23}$$

Despite being dimensional as a result (except for eq. (23)), eq. (20) to eq. (23) serve as corrected performance parameters, considering the effects of different operating conditions in an identical machine with a real gas.

When machines of a different size need to be compared in the plot, those length-dimension variables, neglected in an identical machine, should stay. When machines are using different gases between physical and corrected conditions, gas properties of each machine should be used, respectively.

Compressor corrected maps, for a real gas in an identical machine, can be shown in Fig.3, for example, for corrected pressure ratios. Isentropic efficiency plots are drawn against the corrected mass flow along the corrected speed as well. Fig.4 shows an example of the corrected map of a sample centrifugal compressor using the mixed refrigerant of R-515B, predicted by the meanline program, RCOM1DR ver.7.2. *Please note that the pressure ratio was not corrected yet, but loosely defined as usually accepted in most engineering. To be scientifically valid, it should be corrected using eq. (23).*

Thanks to clear visibility of compressor pressure ratios along the corrected speed still in the unit of rpm over the corrected mass flow still in the unit of kg/s or lbm/s, Group (b) format has been widely used in most of turbomachinery engineering.

Even though some gas industry sectors still use Group (a) format, probably sticking to their tradition, there are many drawbacks in the engineering point of view. First of all, the format demands the value of machine length (D) in advance, which comes inconvenient to engineers. Another drawback is, there is no agreement on the definition of machine length (D) input and speed (N), among those who have been following.

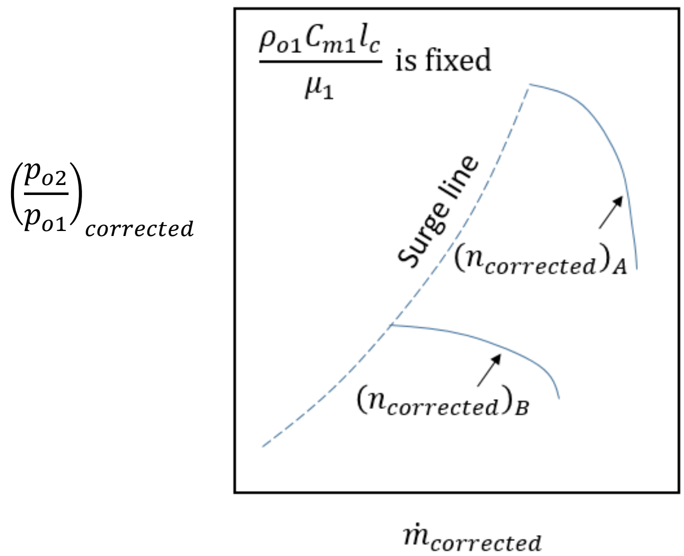

Fig.3 Real-gas compressor corrected map in an identical machine and working gas

Some use the impeller exit diameter for D in the flow coefficient of eq. (7), which will not represent the meaning of suction flow. Some use the rotational speed of “rpm” instead of the inlet sonic velocity in the flow coefficient of eq. (7), which will eventually let the flow coefficient make a U-turn, getting rather smaller, when compressor speed keeps increasing. Despite contradictions to reality, they seem to want to take it for granted. Some use the inlet sonic velocity for ND in the head (or work) coefficient of eq. (3), which will not represent the change of head or work, directly connected with the blade speed (U_2) through the change of angular momentum in the principle of turbomachinery work. Some try to find a compromised way of map presentations through flow coefficient versus pressure ratio, proving that head (or work) coefficient on the normalized map is not engineering-friendly. However, such a format lacks the dimensionless theory after all.

That is why we easily encounter many different definitions of Group (a) parameters, per gas industry, with no common ground among them, leading to uncomfortable inconsistency.

It is therefore strongly recommended to follow Group (b) format in any compressible turbomachinery engineering, especially in aerodynamic design/analysis, interacting with system cycle design/analysis.

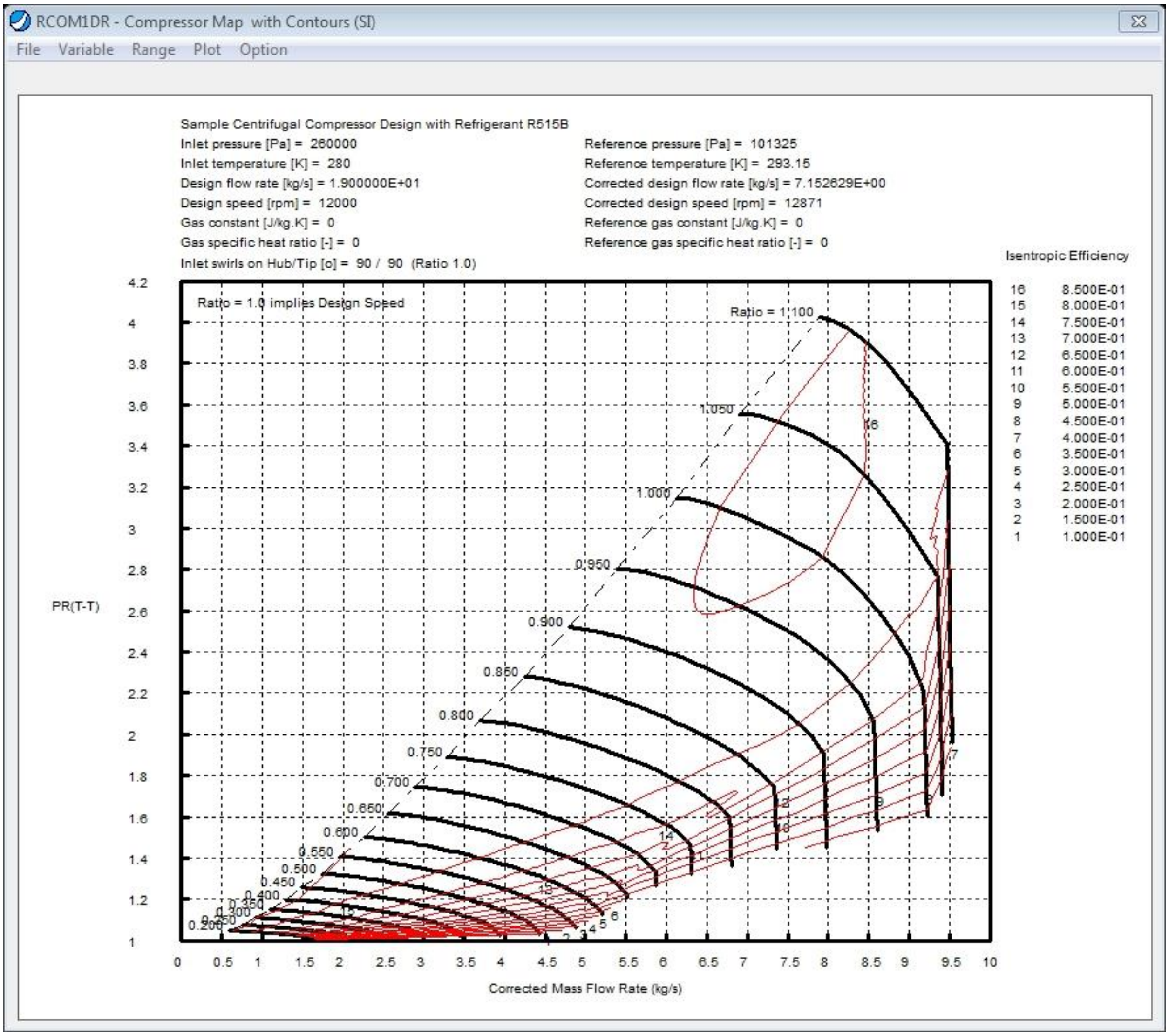

Fig.4 Corrected map of sample centrifugal compressor using real gas of R-515B

predicted by RCOM1DR ver.7.2